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Consolidated Expansions for Estimating the Response of a Randomly Driven Nonlinear Oscillator

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We consider a nonlinear oscillator driven by random, Gaussian "noise." The oscillator, which is damped and has linear and cubic terms in the restoring force, is often called the "Duffing Equation." The Fourier transform of the response is expanded in a series in the coefficient of the nonlinear term. This series is then squared and averaged, and each term in the resulting response spectrum series is expressed in terms of the response spectrum of the linearized harmonic oscillator (i.e., without the cubic term). Since the forcing function is Gaussian, the linear solution is Gaussian. The terms in the series for the response spectrum are then regrouped so that common quantities can be factored out. This process leads to "consolidated equations" for the response spectrum and the "common factors." These consolidated equations are truncated in various ways, and the corresponding solutions are compared with an analog computer experiment. This technique was proposed for turbulent flow by Kraichnan and followed up by Wyld, and has yielded some good results. The numerical results indicate that the truncated consolidated equations can provide a substantial improvement over some other methods used to solve this type of problem. The methods compared with it are (1) the traditional truncated parametric expansion, (2) statistical linearization, and (3) use of the joint-normal hypothesis to express the fourth and sixth moments in terms of the second.

KEY WORDS: random; nonlinear differential equation; nonlinear oscillator; approximation; diagram method; consolidated expansion method.

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1. INTRODUCTION

1.1. Motivation

The usual restrictions used in the theoretical study of turbulence are to an incompressible, Newtonian fluid whose instantaneous-velocity field satisfies the Navier–Stokes and continuity equations. It is supposed that the statistical properties are determined by controllable boundary and/or initial conditions and by the intrinsic properties of the fluid. Only the statistical properties of the velocity field are subjected to prediction and theoretical study. Initial conditions are given statistically.

The principal quantity of interest is the joint probability density functions for the velocity field at many points and many times. We usually focus our attention on the lowest few moments of the two-point, two-time probability density function.

It is well known that if one tries to write down equations for these moments there is always at least one more unknown than equation (e.g., the equation for the second moment contains the third moment, etc.). Any finite collection of these moment equations is therefore indeterminate.

A "theory" for the statistical behavior of such a nonlinear system consists then of a proposal for "closing" the set of moment equations, presumably at finite order. Often this "closure" is effected by a physically motivated hypothesis for expressing the moments of some order in terms of lower moments. In turbulence theory this approach is typified by the spectral transfer (third moment) hypotheses of Kolmogorov and Obukhov, Onsager, Heisenberg, Kovasznay, etc.,[1,2] in terms of the spectrum function (second moment). A different approach has been taken by Millionshtchikov,[3] Tatsumi,[4] Proudman and Reid,[5] Deissler,[6] Kraichnan,[7] and others: the use of more or less formal expansion procedures, with truncation to effect closure. Another approach, pursued by Meecham and Siegel[8] and others[9,10] expands the random-velocity field in terms of "random Hermite polynomials" and achieves closure by truncation.

Up to now the most successful of these, measuring success by agreement with experiment in the absence of empirical constants, has been Kraichnan's "direct interaction approximation."[7] Although it misses slightly one of the most certain properties of large-Reynolds-number turbulence, the Kolmogorov "$-\frac{5}{3}$ law" for the spectrum in the isotropic inertial subrange, it yields good approximations to the spectrum at intermediate Reynolds numbers.[11]

In recent years Kraichnan[12] has concluded that this success of the direct interaction approximation may be fortuitous. Nevertheless it seemed to us worthwhile to test methods of this type on a nonlinear stochastic problem whose results can be got reliably by established methods for comparison. The "oscillator" with linear-plus-cubic "spring" was picked.

It has become clear that Kraichnan's direct-interaction approximation is a special truncation of an expansion of the turbulent-velocity field in powers of the Reynolds number. Since turbulence occurs only for Reynolds numbers larger (usually much larger) than unity, Kraichnan[7] pointed out that this expansion probably diverges everywhere. His first truncation retains terms of all powers in the expansion,

though an ever decreasing fraction of those of increasingly high order. The nature of this class of truncations is perhaps best displayed by the use of diagrams to represent the physical quantities and operations.[3]

These diagrams, introduced to turbulence theory by Kraichnan,[13] are convenient for the identification of groups of factors with common properties among classes of terms in the expansions. This suggests the regrouping of the terms into (infinite) subsets such that these groups of factors can be "isolated." Each group can then be represented by a single symbol, giving the expansion expressed as a sequence of "consolidated diagrams." It is these infinite consolidated sequences which are then truncated.

Kraichnan[7] was also the first to propose and identify explicitly equations which were a formal summation, in the sense of the paragraph above, of an infinite group of terms from the Reynolds-number expansion. By restricting his attention to a particular class of initial conditions, he developed two simultaneous equations for the energy spectrum and the "infinitesimal-impulse response function," a kind of generalized Green's function. One truncation of these equations is his "direct interaction approximation," a pair of integral equations when written in traditional symbols.

He has shown that his direct interaction approximation equations are exact for a "model" system.[13] This assures that realizability conditions, e.g., nonnegative definite energy spectrum, are satisfied. Numerical results turn out remarkably well for the case of isotropic turbulence.[11]

In a more recent paper, Kraichnan[12] stated that his second-order approximation[4] is not as good as his first (direct interaction) approximation. Later he introduced a somewhat different method of consolidation by adding the generalized "vertex operator" concept.[14] This results in three simultaneous equations for three unknowns, and permits different truncations. We have not encountered any turbulence computations using the generalized vertex operator.[15]

Shortly thereafter H. W. Wyld[16] put forth a more systematic exposition of the diagrammatic approach. We have employed the Wyld procedures more or less closely in testing the method on the Duffing equation. Truncations involving all three generalized equations are designated here as "Kraichnan–Wyld approximations."

1.2. The Duffing Equation

Because of the complexity of the Navier–Stokes equations, and because of the *ad hoc* character of the truncation of the consolidated expansions, it is very difficult to evaluate *a priori* the "approximations" mentioned above. Their consequences can, however, be compared with experiment. In this paper we consider a much

[3] The alphabetical symbols representing physical quantities, and the plus, minus and summation signs representing operations, are also diagrams.

[4] The direct-interaction approximation involves expanding the third moment which appears in the equation for the second moment, in a Reynolds-number series, and consolidating in a particular way. Higher approximations involve expanding higher moments in Reynolds-number series and consolidating.

simpler nonlinear random system, a damped oscillator with a cubic term in the restoring force, driven by a random forcing function:

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + x + \beta x^3 = f(t) \tag{1}$$

With $f = 0$ this is often called the Duffing equation.[17] The study will be confined to cases in which $f(t)$ is a Gaussian random process.

Simplicities of this equation relative to the Navier–Stokes equation for testing truncations include: (a) that it has fewer variables, (b) that its statistical response can be studied *via* a Fokker–Planck equation,[18] and (c) that the coefficient β of the nonlinear term can be meaningfully set at a continuous range of values beginning at zero; i.e., there exists a range of "small nonlinearity." It is significant to note, however, that a β-expansion around $\beta = 0$ does not converge. This is important in light of the fact that Kraichnan has speculated that the Reynolds number expansion of the solution of the Navier–Stokes equation does not converge for Reynolds numbers where turbulence can be supported. Since (1) is clearly well-behaved for $\beta > 0$, it may be true that a β-series would converge in some region if the expansion were around any positive β, say β_0 (certainly having a radius of convergence less than β_0). A β-expansion around $\beta = 0$ would have to converge for $\beta < 0$ as well as $\beta > 0$. For $\beta < 0$, a large enough "kick" from the forcing function could put the response into a region where the "effective spring constant" is negative (i.e., $|\beta|\,|x^3| > |x|$). The response would then continue to grow. For a Gaussian forcing function, the probability of a large enough kick is finite so the average properties of the response are not defined in the limit of large time.

In Section 3 we develop a set of consolidated equations for the Duffing equation along the lines of Wyld's work. These are then truncated in several different ways and solved numerically. We try to interpret these truncations in terms of the approximations mentioned in Section 1.1.

We can test the results in two ways. First, the Fokker–Planck equation for the steady-state, one-time, velocity–displacement, joint probability density function can be solved,[19] and hence we can get good approximations to mean-square displacement (second moment), third moment, fourth moment, etc.[5] Second, an analog-computer experiment gives the form of the power spectrum for a range of governing parameters. We have found no published data on measurements of statistical quantities for the Duffing equation.[20]

2. USEFUL AUXILIARY RESULTS

2.1. The Steady State One-Point Probability Density Function for the Duffing Equation

If we postulate a delta-correlated Gaussian forcing function, the Fokker–Planck equation for the response probability density function is exact. It therefore provides

[5] For an idealized system we can postulate a Dirac-correlated forcing function ("white noise"), in which case the Fokker–Planck result is exact.

a good standard against which to test the various approximate truncation methods. The Fokker–Planck treatment is discussed in References 18 and 19. The resulting steady state, onepoint joint probability density for the response x and its time derivative $y \equiv dx/dt$ is

$$P_{xy}(u, v) = C \exp\left\{-\left[\frac{\alpha v^2}{D} + \frac{2\alpha}{D}\left(\frac{u^2}{2} + \frac{\beta u^4}{4}\right)\right]\right\} \tag{2}$$

where C is chosen to normalize[6] $P_{xy}(u, v)$. Because P_{xy} is not integrable in closed form, we cannot write down an analytic expression for C. D is a statistical property of the forcing function, essentially its "diffusivity."[18]

We see from Eq. (2) that the time derivative of the response is Gaussian. The probability density function of x alone follows by integration over v from $-\infty$ to ∞:

$$P_x(u) = C \exp\left\{-\frac{2\alpha}{D}\left(\frac{u^2}{2} + \frac{\beta u^4}{4}\right)\right\} \tag{3}$$

[6] Note that for $\beta < 0$, P_{xy} is not integrable.

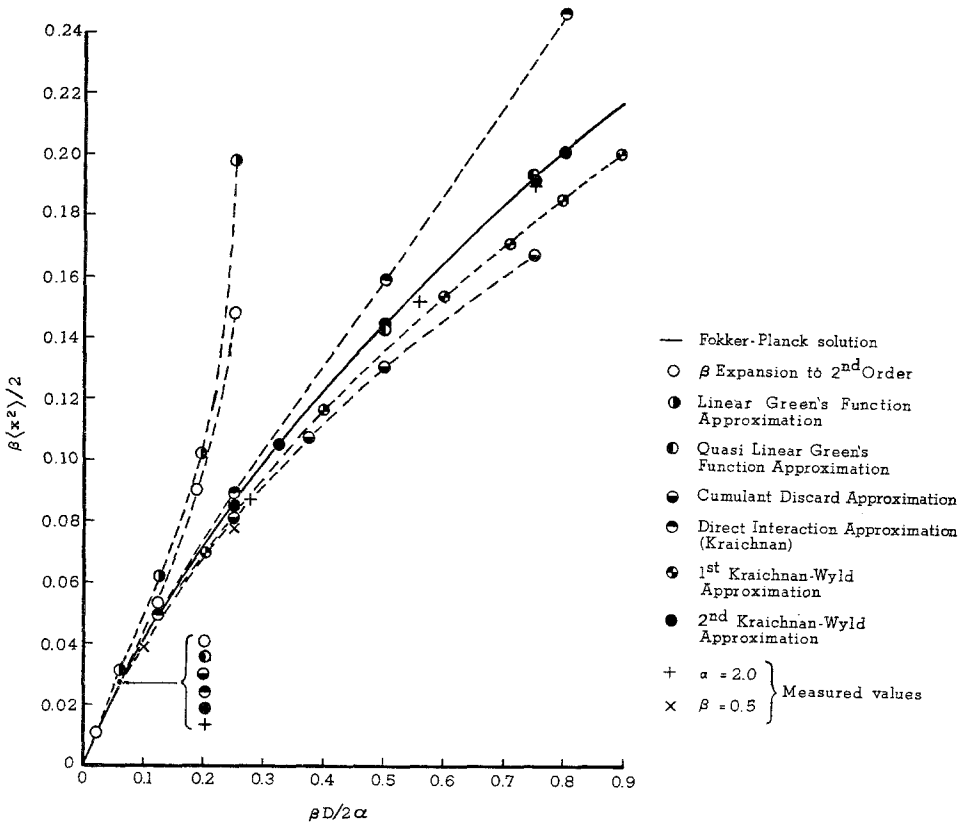


Fig. 1. Mean-square response as a function of $\beta D/2\alpha$, a parameter measuring roughly the amount of nonlinearity effect.

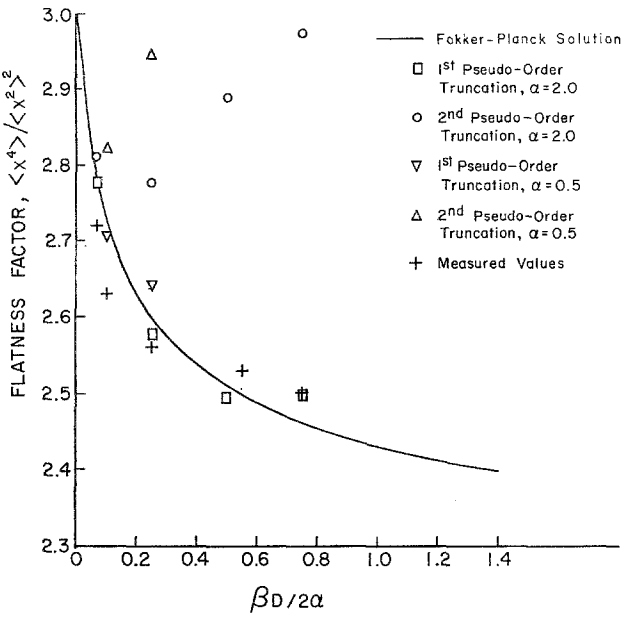


Fig. 2. Flatness factor (kurtosis) of response as a function of $\beta D/2\alpha$.

We find functions[7]

$$\frac{\beta}{2}\langle x^2\rangle = L\left(\frac{\beta D}{2\alpha}\right) \tag{4}$$

and

$$\frac{\langle x^4\rangle}{\langle x^2\rangle^2} = A\left(\frac{\beta D}{2\alpha}\right) \tag{5}$$

These two functions are plotted in Figs. 1 and 2.

2.2. The Analog Computer Simulation

The Fokker–Planck analysis mentioned above was confined to single-time probabilities, but we are very much interested in how well the truncated expansions approximate the power spectrum, a two-time function because it is the Fourier transform of the autocovariance. Therefore experiments were done on an electronic system which behaved essentially like Eq. (1).[18,23]

The measured output second and fourth moments are included in Figs. 1 and 2 respectively, and the measured output power spectra are included in Figs. 10–12. Spectrum measurements were made by feeding the analog computer output through an adjustable bandpass filter (Dytronic, Model 720, medium bandwidth), squaring and averaging.

[7] $\langle\ \rangle$—indicate ensemble average.

The nonzero bandwidth of any real filter causes a systematic "error" which can in principle be "corrected" by solving an integral equation. This is rather complex, so we followed instead the inverse procedure of subjecting all theoretical spectra to a numerical "filtering" process with the same bandpass function as was imposed by the spectral analyzer in the experiment. Both filtered and unfiltered theoretical spectra are included in each comparison.

3. THE CONSOLIDATED EQUATIONS

3.1. Introduction

In this section we develop two kinds of consolidated equations for the power spectrum of the solution of Eq. (1).

We first develop a pair of equations for the power spectrum and a "generalized (modified) Green's function."[8] These correspond to Kraichnan's earlier work. Although none of the truncations tried here corresponds precisely to his "direct interaction approximation," one of them is roughly the same order—as will be discussed.

We then develop a slightly different form of consolidation, involving the use of a "generalized (modified) vertex operator" like that suggested for turbulence by Kraichnan[21] some time after his earliest papers. This approach involves a total of three consolidated equations of infinite order in the three unknown functions. We use a formalism like that of Wyld.

In order to develop these consolidated expansions, we first consider a (traditional) β-expansion of the Fourier transform of Eq. (1). We then devise a set of diagrams which can be put into one-to-one correspondence with the terms in this expansion. A correspondence between the process of averaging terms in the traditional parametric expansion and a process of combining diagrams will be established. It can then be shown that the further combining ("consolidation") of structurally related groups of diagrams is equivalent to summing over infinite subsets of terms in the parametric expansion. Each group of structurally analogous terms is then represented as a consolidated diagram in the manner of Kraichnan[21] and Wyld,[16] giving the consolidated equations.

3.2. Parametric Expansion of the Fourier Transform of the Response of the Duffing Oscillator Equation

We start with the Duffing oscillator, Eq. (1) with $f(t)$ restricted to be a Gaussian random function.[9] For further simplicity, we consider only the statistically stationary state. We could let

$$u(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} x(t)\, dt$$

[8] There is no standard terminology for these functions. We use that of Lee. [15]

[9] Cf. Section 3.4.

Since $x(t)$ is not square-integrable, this integral doesn't exist. We could, however, use it in the sense of a generalized function.[19,22]

Instead we will assume $x(t)$ is periodic with period $2T$ and let $T \to \infty$ in the final equations. This formulation has been used by Heisenberg, Kraichnan, Wyld, and others in the turbulence problem.

Let

$$x(t) = \sum_{\omega} u(\omega)\, e^{i\omega t}$$

$$f(t) = \sum_{\omega} g(\omega)\, e^{i\omega t} \tag{6}$$

where

$$u(\omega) = \frac{1}{2T} \int_{-T}^{T} x(t)\, e^{-i\omega t}\, dt$$

and

$$g(\omega) = \frac{1}{2T} \int_{-T}^{T} f(t)\, e^{-i\omega t}\, dt$$

Since x is real, $u(\omega) = u^*(-\omega)$.[10]

In the limit $T \to \infty$, $\sum_{\omega} \to 1/2\pi \int_{-\infty}^{\infty} d\omega$.

The Fourier transform of Eq. (1.2.1) is

$$u(\omega)\{1 - \omega^2 + i\alpha\omega\} + \beta \sum_{\omega'} \sum_{\omega''} u(\omega - \omega')\, u(\omega' - \omega'')\, u(\omega'') = g(\omega) \tag{7}$$

The sum in Eq. (7) is equivalent to

$$\sum_{\omega_1} \sum_{\omega_2} \sum_{\omega_3} u(\omega_1)\, u(\omega_2)\, u(\omega_3)$$

with the triple sum restricted to frequencies such that $\omega_1 + \omega_2 + \omega_3 = \omega$. It is effectively a double sum, now written for brevity as

$$\sum u(\omega_1)\, u(\omega_2)\, u(\omega_3); \qquad \omega_1 + \omega_2 + \omega_3 = \omega$$

We shorten the notation a little further by writing the subcondition

$$\omega_1 + \omega_2 + \omega_3 = \omega_4 \leftrightarrow (1, 2, 3 \mid 4)$$

where $\omega \equiv \omega_0$ for the writing of this condition. Conditions such as $\omega_1 + \omega_2 = \omega_3 + \omega_4$ will be written as $(1, 2 \mid 3, 4)$, etc. In our shortened notation Eq. (7) is

$$u(\omega)\{1 - \omega^2 + i\alpha\omega\} + \beta \sum_{(1,2,3\mid 0)} u(\omega_1)\, u(\omega_2)\, u(\omega_3) = g(\omega). \tag{7'}$$

[10] (*) indicates complex conjugate.

If $\beta = 0$, Eq. (1) is just the linear, damped harmonic oscillator. Let $u_0(\omega)$ be the Fourier transform of the response of this linear system:

$$u_0(\omega) = S(\omega)\, g(\omega) \tag{8}$$

where

$$S^{-1}(\omega) = 1 - \omega^2 + i\alpha\omega. \tag{9}$$

We now write u in Eq. (7) as a series in powers of β:

$$u(\omega) = u_0(\omega) + u_1(\omega) + u_2(\omega) + \cdots \tag{10}$$

The subscript indicates the order of β involved. We do not follow the usual convention of writing the expansion as an explicit power series in β, e.g.,

$$u = u_0 + \alpha_1\beta + \alpha_2\beta^2 + \cdots$$

because the forms of the terms $u_j(\omega)$ are conveniently checked with β's dispersed as internal factors, as will be seen in Section 3.3. We substitute Eq. (10) into Eq. (7) and equate separately to zero terms of the same order in β.[11]

$$u_1(\omega) = -S(\omega)\,\beta \sum_{(1,2,3|0)} u_0(\omega_1)\, u_0(\omega_2)\, u_0(\omega_3) \tag{11}$$

$$u_2(\omega) = -3S(\omega)\,\beta \sum_{(1,2,3|0)} u_0(\omega_1)\, u_0(\omega_2)\, u_1(\omega_3) \tag{12}$$

$$u_3(\omega) = -3S(\omega)\,\beta \sum_{(1,2,3|0)} \{u_0(\omega_1)\, u_0(\omega_2)\, u_2(\omega_3) + u_0(\omega_1)\, u_1(\omega_2)\, u_1(\omega_3)\} \tag{13}$$

etc.

By replacing each u_k on the right sides in terms of lower order terms, we can express each term in the series for u_n in terms of u_0 only:

$$u_1 = -S(\omega)\,\beta \sum_{(1,2,3|0)} u_0(\omega_1)\, u_0(\omega_2)\, u_0(\omega_3) \tag{14}$$

$$u_2 = 3S(\omega)\,\beta \sum_{(1,2,3|0)} [u_0(\omega_1)\, u_0(\omega_2)\, S(\omega_3)\,\beta \sum_{(4,5,6|3)} [u_0(\omega_4)\, u_0(\omega_5)\, u_0(\omega_6)] \tag{15}$$

$$\begin{aligned} u_3 = {} & -3S(\omega)\,\beta \sum_{(1,2,3|0)} \Big\{ u_0(\omega_1)\, u_0(\omega_2)\, 3S(\omega_3)\,\beta \\ & \times \sum_{(4,5,6|3)} [u_0(\omega_4)\, u_0(\omega_5)\, \beta S(\omega_6) \sum_{(7,8,9|6)} u_0(\omega_7)\, u_0(\omega_8)\, u_0(\omega_9)] \Big\} \\ & - 3\beta S(\omega) \sum_{(1,2,3|0)} \Big\{ u_0(\omega_1)\, S(\omega_2)\, S(\omega_3)\,\beta \\ & \times \sum_{(4,5,6|2)} [u_0(\omega_4)\, u_0(\omega_5)\, u_0(\omega_6)]\,\beta \sum_{(7,8,9|3)} u_0(\omega_7)\, u_0(\omega_8)\, u_0(\omega_9) \Big\} \end{aligned} \tag{16}$$

etc. Here the orders in β are clear: $u_j \sim \beta^j$.

[11] It should be noted that there is no relation between the subscripts of u_n and ω_p.

3.3. Diagrammatic Representation of the Terms in the Parametric Expansion for $u(\omega)$

It is useful to introduce a diagrammatic representation for the terms in the series for u for three reasons. First, it is much easier to write down the diagrams for the higher order terms than to write out the terms themselves. Second, these diagrams prove to be a viable language for expressing equations for the statistical properties of u. Third, they are heuristically valuable for combining terms according to their structural properties, to form consolidated series.

The diagrammatic symbols will be

straight line $(\text{——}) \leftrightarrow S(\omega)$

dot "vertex" $(\cdot) \leftrightarrow -\beta$

dashed line $(- - -) \leftrightarrow u_0(\omega)$.

The straight line is the Green's function for the linearized ($\beta = 0$) case.

The diagrams for the first six terms in the series for u are shown in Fig. 3.

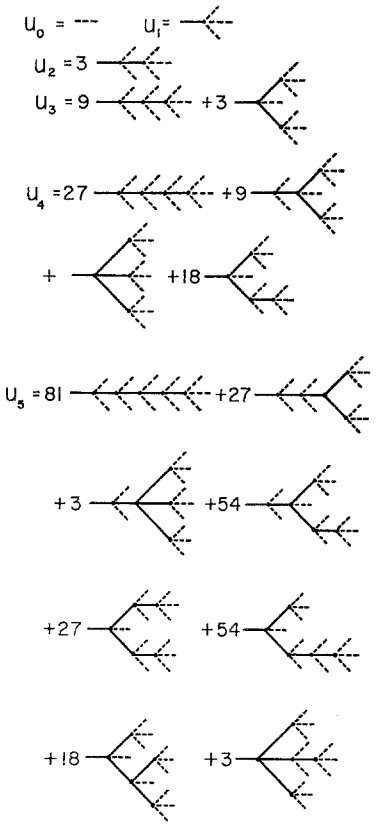


Fig. 3. Diagrams for the terms in the expansion for corresponding to Eqs. (14), (15), and (16) and similar equations for higher-order terms.

In order to establish a one-to-one correspondence, we set down rules for forming these diagrams.

Rule 1. Four elements are joined at each vertex, at least one of which is a straight line.

Rule 2. Frequency is conserved at each vertex: the sum of the frequencies of the elements to the right of a vertex equals that of the elements to the left. Summation is implied over all frequencies, consistent with the above conservation.

Rule 3. There is a factor of 3 associated with every vertex which has only 1 or 2 dashed lines entering it.

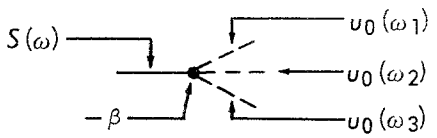
Rule 4. When the total number of solid lines to the right of a trifurcation with two solid lines is not the same along the two paths following these solid lines, multiply the diagram by a factor of 2. Start with the trifurcation furthest to the right and apply this test, then work to the left along all possible paths. For trifurcations yielding three solid lines we must consider three possible cases. If the three branches have the same number of solid lines, there is no coefficient. If two branches have the same number of solid lines, multiply the diagram by a factor of 3, otherwise, multiply by 6.

Rule 5. There are no closed polygons.

Rule 6. The complex conjugate of a diagram is its mirror image in a vertical plane.

Rules 1–5 follow directly from examination of the form of the terms in (10). Rule 1 comes from the cubic form of the nonlinear term. Rule 2 comes from the summations. Rules 3 and 4 come from the symmetry of the terms in the sum. Rule 5 follows from the fact that there are no sums of wavenumbers adding to zero.

Consider now $u_1 = -S(\omega)\beta \sum_{(1,2,3|0)} u_0(\omega_1)\, u_0(\omega_2)\, u_0(\omega_3)$. In terms of diagrams this is



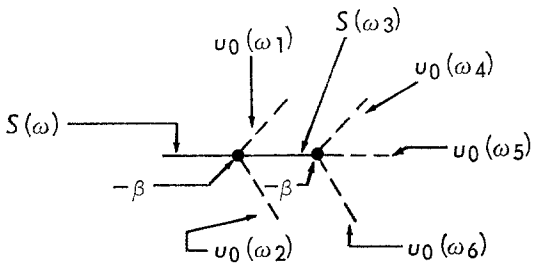
The corresponding analytical expression is the product of the elements in the diagram. A summation over all ω_1, ω_2, and ω_3 such that $\omega_1 + \omega_2 + \omega_3 = \omega$ is implied by rule 2.

Next consider

$$u_2 = -3S(\omega)\,\beta \sum_{(1,2,3|0)} u_0(\omega_1)\, u_0(\omega_2)\, u_1(\omega_3)$$

$$= 3S(\omega)\,\beta \sum_{(1,2,3|0)} \left[u_0(\omega_1)\, u_0(\omega_2)\, S(\omega_3)\,\beta \sum_{(4,5,6|3)} u_0(\omega_4)\, u_0(\omega_5)\, u_0(\omega_6) \right]$$

In terms of diagrams[12] this becomes



The product of these elements is $S(\omega)\,\beta u_0(\omega_1)\,u_0(\omega_2)\,S(\omega_3)\,\beta u_0(\omega_4)\,u_0(\omega_5)\,u_0(\omega_6)$. A summation over ω_1, ω_2, ω_3, ω_4, ω_5, ω_6 such that $\omega_1 + \omega_2 + \omega_3 = \omega$ and $\omega_4 + \omega_5 + \omega_6 = \omega_3$ is required by Rule 2. By writing down a few terms in the series and their corresponding diagrams, it can be seen that there is a one-to-one correspondence.

To draw u_n, take n Green's functions (straight lines) and n vertices; combine them in all possible ways consistent with the above rules and add the necessary $(2n + 1)$ u_0's (dashed lines).

3.4. Construction of Diagrams for the Terms in the Equation for the Response Spectrum

The series (10) involves random quantities, but we are interested in the averaged properties of response or of its Fourier amplitude $u(\omega)$.

To get a "simple" equation for the spectrum $\langle u(\omega)\,u(-\omega)\rangle$,[13] we restrict the forcing function $f(t)$ to be a Gaussian random process with zero mean. Then all of its statistical properties can be given in terms of its second moment. If f were a general random function, all of the moments of f would have to be given.[14]

We shall also limit our study to the stationary response of Eq. (1). If $x(t)$ is a stationary random function, its covariance

$$\langle x(t_1)\,x(t_2)\rangle = R(t_2 - t_1) \tag{17}$$

is a function of $(t_2 - t_1)$ only. From Eq. (6)

$$\langle x(t_1)\,x(t_2)\rangle = \sum_{\omega'}\sum_{\omega''} \langle u(\omega')\,u(\omega'')\rangle \exp[i(\omega' t_1 + \omega'' t_2)]$$

[12] We adopt the arbitrary convention of drawing all diagrams as symmetrically as possible. In particular, each trifurcation is made symmetric. In order to conform with Rule 2, care must be taken to keep elements on the proper (right–left) side of the vertex.

[13] $< >$—indicate ensemble average.

[14] A case of intermediate complexity is that with f derived from a Gaussian process by a zero-memory nonlinear transformation. In that case the higher moments can still be expressed in terms of the second moment.

If Eq. (17) is to hold, $\omega' = -\omega''$ so

$$\langle u(\omega')\, u(\omega'')\rangle = U(\omega')\, \delta_{\omega',-\omega''} \tag{18}$$

where U is the spectrum function and

$$\delta_{\omega_1,\omega_2} \equiv \begin{cases} 1 & \text{if } \omega_1 = \omega_2 \\ 0 & \text{if } \omega_1 \neq \omega_2 \end{cases}$$

Since a linear transformation of a Gaussian random process yields a Gaussian random process, $g(\omega)$ is Gaussian (the Fourier transform being linear). Thus $u_0(\omega)$, a linear function of $g(\omega)$, is also Gaussian.

By stationarity

$$\langle u_0(\omega_1)\, u_0(\omega_2)\rangle = U_0(\omega_1)\, \delta_{\omega_1,-\omega_2}$$

and, because of the Gaussian behavior,

$$\begin{aligned}\langle u_0(\omega_1)\, u_0(\omega_2)\, u_0(\omega_3)\, u_0(\omega_4)\rangle = {} & \delta_{\omega_1,-\omega_2}\delta_{\omega_3,-\omega_4} U_0(\omega_1)\, U_0(\omega_3) \\ & + \delta_{\omega_1,-\omega_3}\delta_{\omega_2,-\omega_4} U_0(\omega_1)\, U_0(\omega_2) \\ & + \delta_{\omega_1,-\omega_4}\delta_{\omega_2,-\omega_3} U_0(\omega_1)\, U_0(\omega_2)\end{aligned} \tag{19}$$

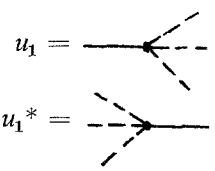
We get similar results for all higher even moments; all odd moments of U_0 are zero.

To get an equation for the spectrum of the response, we multiply the series (10) for $u(\omega)$ by the series for $u(-\omega) = u^*(\omega)$.

Write $\langle u(\omega)\, u(-\omega)\rangle \equiv U(\omega)$; then

$$\begin{aligned}U(\omega) = {} & \langle u_0(\omega)\, u_0{}^*(\omega)\rangle + \langle u_0(\omega)\, u_1{}^*(\omega)\rangle + \langle u_1(\omega)\, u_0{}^*(\omega)\rangle + \langle u_0(\omega)\, u_2{}^*(\omega)\rangle \\ & + \langle u_1(\omega)\, u_1{}^*(\omega)\rangle + \langle u_2(\omega)\, u_0{}^*(\omega)\rangle + \cdots\end{aligned} \tag{20}$$

We recall, by rule 6, that the complex conjugate of a diagram is its mirror image in a vertical plane, e.g.,



From the diagrams in the series for u, we now synthesize a set of diagrams that are in one-to-one correspondence with the terms in the series for $U(\omega)$.

To do this we must introduce a new symbol. Let

$$U_0 = \langle u_0 u_0{}^*\rangle \equiv \langle \text{---} \quad \text{---} \rangle = \sim\sim$$

To show how diagrams for $U(\omega)$ are formed, first consider the term $\langle u_1(\omega)\, u_1(-\omega)\rangle$ in Eq. (20). Analytically this corresponds to

$$\left\langle S(\omega)\, \beta \sum_{(1,2,3\,|\,0)} [u_0(\omega_1)\, u_0(\omega_2)\, u_0(\omega_3)] \cdot S(-\omega)\, \beta \sum_{(4,5,6\,|\,0)} [u_0{}^*(\omega_4)\, u_0{}^*(\omega_5)\, u_0{}^*(\omega_6)] \right\rangle$$

Using the sixth-order Gaussian condition analogous to (19),

$$\langle u_1(\omega)\, u_1(-\omega)\rangle = 6S(\omega)\,\beta \sum_{(1,2,3|0)} \{U_0(\omega_1)\, U_0(\omega_2)\, U_0(\omega_3)\}\,\beta S(-\omega)$$

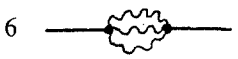
(a)

$$+\, 9S(\omega)\,\beta \left\{\sum_{\omega'} U_0(\omega')\right\} U_0(\omega) \left\{\sum_{\omega''} U_0(\omega'')\right\} \beta S(-\omega) \qquad (21)$$

(b)

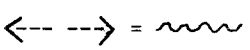
We see that the process is to take all the u_0, combining them in pairs in all possible combinations and keeping track of the summations involved.

In terms of diagrams, term (a) above is



and (b) above is

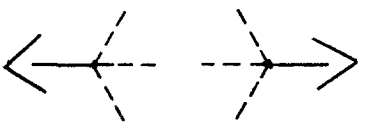
To form these diagrams, we place the two "tree-like" diagrams corresponding to the two u_n's in positions such that the dotted lines face each other, then we combine dotted lines in all possible pairs, using the relation



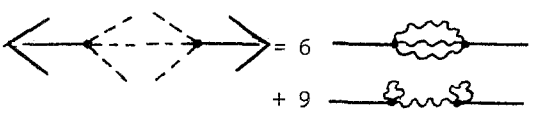
Note that <--> = 0.

In terms of diagrams,

$\langle u_1(\omega)\, u_1(-\omega)\rangle =$



Combining u_0's as we did in the analytic form is equivalent to combining pairs of dotted lines to give wavy lines. Thus

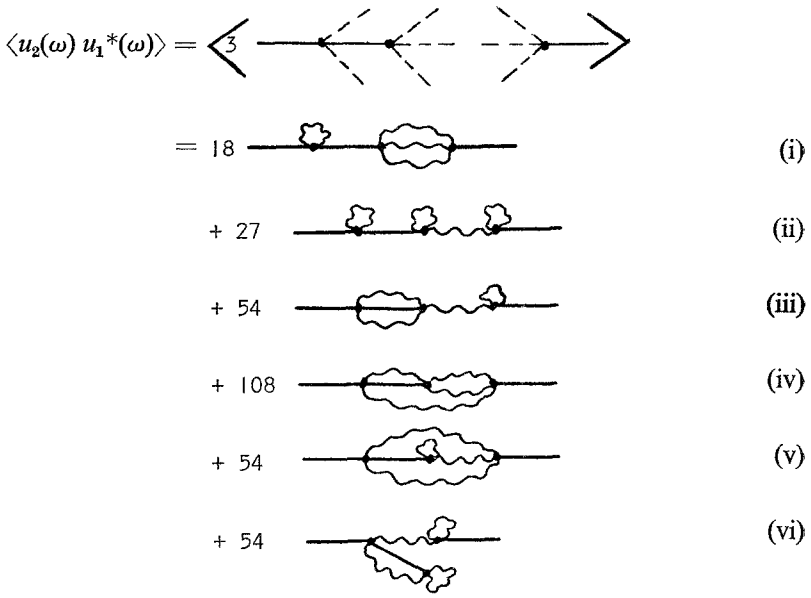


(22)

Identical[15] diagrams result from permutations of dummy variables in the summation. These are mathematically identical.

[15] There is no unique way to draw these diagrams. Thus, two diagrams which *can* be drawn to look identical must be considered identical.

A more complicated example is one of the third-order terms:



(23)

Each of these diagrams represents a product of the factors involved (e.g., S, U_0, or β). Since these diagrams are terms in the series representation of $\langle u(\omega)\, u(-\omega)\rangle$, it follows that the first and last elements are evaluated at frequencies ω and $-\omega$ respectively. Consider for example

The rest of the elements are given "dummy" frequencies. In the above example, the three U_0 factors are given "dummy" frequencies, say ω_1, ω_2, ω_3. Then the product of factors in the above diagram becomes:

$$S(\omega)\,\beta U_0(\omega_1)\,U_0(\omega_2)\,U_0(\omega_3)\,\beta S(-\omega).$$

Rule 2 states that we must sum over all dummy indices with the condition that frequency is conserved at each vertex. The summations are not associated with the vertices—they are associated with the dummy frequencies. Only the conditions on the summations are associated with the vertices.[16]

[16] Every diagram in the series for $U(\omega)$, Fig. 4, can be cut in two by passing a vertical line through it that intersects only U_0's. All S to the right of this line are complex conjugates of those to the left of the line. Note that $S^*(\omega) = S(-\omega)$. In order to apply rule 2 consistently, we must recall that $U_0(\omega) = \langle u_0(\omega) u_0(-\omega)\rangle$ and, hence, U_0 is a function of both ω and $-\omega$. The ω argument is then associated with the vertex that U_0 enters from the left and the $(-\omega)$ argument is associated with the vertex (which may be the same vertex) that it enters from the right.

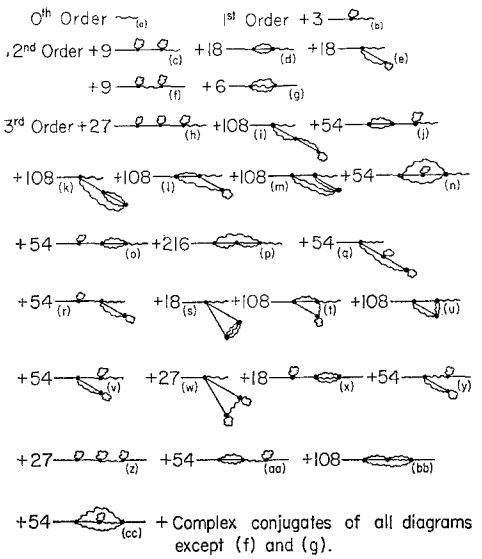


Fig. 4. Diagrams for the spectrum function $U(\omega)$ defined in (20) to third order in β.

Thus in the example above we must sum over all ω_1, ω_2, and ω_3 subject to the following conditions: the vertex on the left implies that $\omega_1 + \omega_2 + \omega_3 = \omega$ and the vertex on the right implies that $(-\omega_1) + (-\omega_2) + (-\omega_3) = (-\omega)$. Of course these conditions are identical, so that the diagram in the example above becomes

$$S(\omega)\,\beta \sum_{(1,2,3|0)} \{U_0(\omega_1)\, U_0(\omega_2)\, U_0(\omega_3)\}\, \beta S(-\omega)$$

Consider the following examples.

1) 

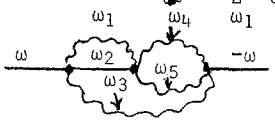
The product of factors becomes

$$-S(\omega)\,\beta U_0(\omega_1)\, S(+\omega).$$

Rule 2 implies a summation over ω_1. The condition at the vertex is $\omega + \omega_1 = \omega_1 + \omega$. Thus we get:

$$-\beta S(\omega) \left\{ \sum_{\omega_1} U_0(\omega_1) \right\} S(\omega)$$

Note that  $= \sum_{\omega_1} U_0(\omega_1) = \langle x_0^2 \rangle$.

2) 

The frequency associated with each element is shown. The product of factors in this diagram becomes

$$-S(\omega)\,\beta U_0(\omega_1)\,S(\omega_2)\,\beta U_0(\omega_3)\,U_0(\omega_4)\,U_0(\omega t)\,\beta S(-\omega)$$

We must sum over all ω_1, ω_2, ω_3, ω_4, and ω_5 subject to the following conditions at the vertices: (i) the leftmost vertex implies $\omega_1 + \omega_2 + \omega_3 = \omega$; (ii) the center vertex implies that $\omega_4 + \omega_5 = \omega_1 + \omega_2$; (iii) the rightmost vertex implies that $-\omega_4 - \omega_5 - \omega_3 = -\omega$. This last condition is redundant. Thus we get

$$-S(\omega)\,\beta \sum_{(1,2,3\,|\,0)} U_0(\omega_1)\,S(\omega_2)\,U_0(\omega_3)\,\beta \sum_{(4,5\,|\,1,2)} \{U_0(\omega_4)\,U_0(\omega_5)\}\,\beta S(-\omega)$$

Diagrams for all the terms to third order in β in the $U(\omega)$ expansion are shown in Fig. 4.[17] Complex conjugates have been omitted.[18]

3.5. Construction of the Consolidated Equations

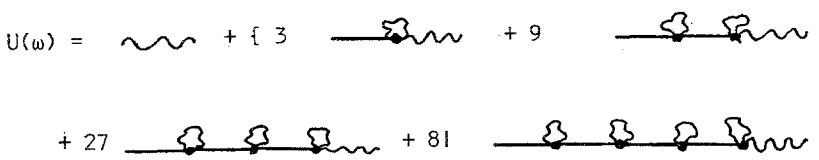
We have now developed a β-expansion for the power spectrum of the Duffing equation response to a Gaussian forcing function, and we have devised a set of diagrams to represent the terms in this expansion. The next major step is consolidation of this expansion into a more compact representation.

Consolidation is a process of collecting the terms into infinite subgroups, each represented by a single symbol. Since each consolidated term contains an infinite subset of the original terms, each contains terms of all orders in β.

A feature of the Duffing and Navier–Stokes consolidations is that some of the consolidated elements turn out to be expressible in terms of others [see Eq. (30)]. This means we shall not need a new symbol for each consolidation. In these cases the consolidation process transforms an explicit expression for the spectrum $U(\omega)$ (with all quantities on the right side known) back into an equation for $U(\omega)$, with this unknown function appearing on both sides of the equation.

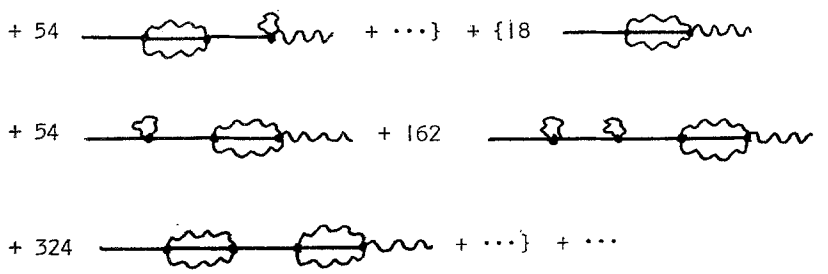
For the diagrams in Fig. 4, it is convenient to form three consolidated equations, one for the spectrum function itself, one for a "generalized Green's function," and one for a "generalized vertex operator" (to be explained in more detail later).

Looking at Fig. 4 for terms with common factors, we see, for example,



[17] Fourth-order diagrams are available.[23]

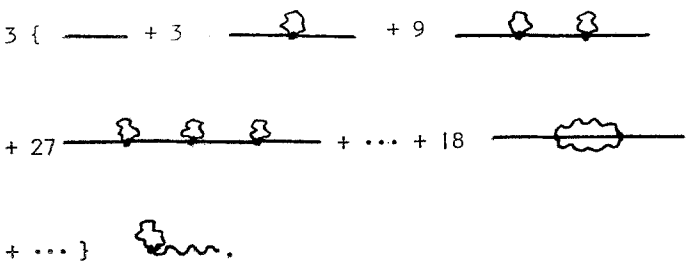
[18] The complex conjugate of a diagram is its mirror image in a vertical plane so that symmetric diagrams are real. Thus to make Fig. 4 complete we must add to it mirror images of all unsymmetric diagrams.



(24)

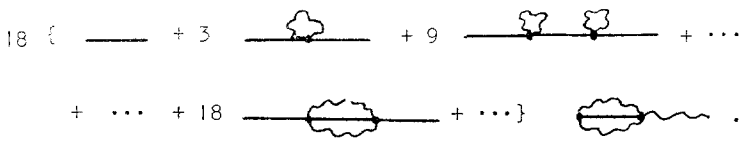
Here the relevant terms $\sim\beta^4$ have been included.

The first group of terms in curly brackets can be factored as



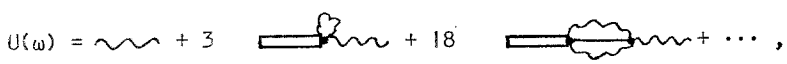
(25)

The second group of terms in curly brackets in (24) can be factored as



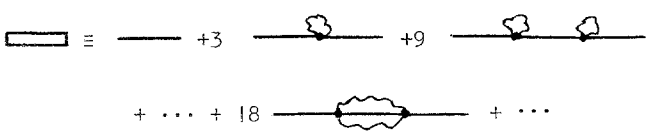
(26)

In (25) and (26) the terms appearing in the curly brackets are identical. Furthermore, factors attached to the ends of these terms cannot distinguish among them.[19] Thus we can represent each group of diagrams in (24) as one diagram, introducing a new symbol for this "generalized Green's function." Then (24) becomes



(27)

where we define



(28)

The second term on the right of series (27) is a consolidation of the terms in series (25) and the third term a consolidation of the terms in the series (26).

[19] We might say that their "input" and "output" effects are identical with those of a simple Green's function $S(\omega)$, i.e., ——.

A similar consolidation can be effected, for example, in the following groups in Fig. 4:

$$\{(e) + (r) + \cdots\}; \{(g) + (x) + \cdots\}.$$

We shall represent ⊏⊐, the generalized Green's function, by the letter $S'(\omega)$.

The characteristics which guarantee that an element[20] is included in the generalized Green's function are:

(1) the element begins and ends with a straight line,

(2) a straight line runs completely through it.

All elements to third order with these characteristics are shown in Fig. 5.

To see why these characteristics are important for consolidation, pick a diagram in Fig. 4, say diagram (g). This has two simple Green's functions (——). If we replace either one of these, say the one on the left, by any element in Fig. 5, say (b), we get another diagram in Fig. 4: we get (x).[21]

[20] By an element, we mean a part of a diagram, e.g., a factor in the corresponding analytical expression.

[21] Note that if we take an element with either characteristic above but not both and replace a single Green's function by it we do not get another diagram in Fig. 4.

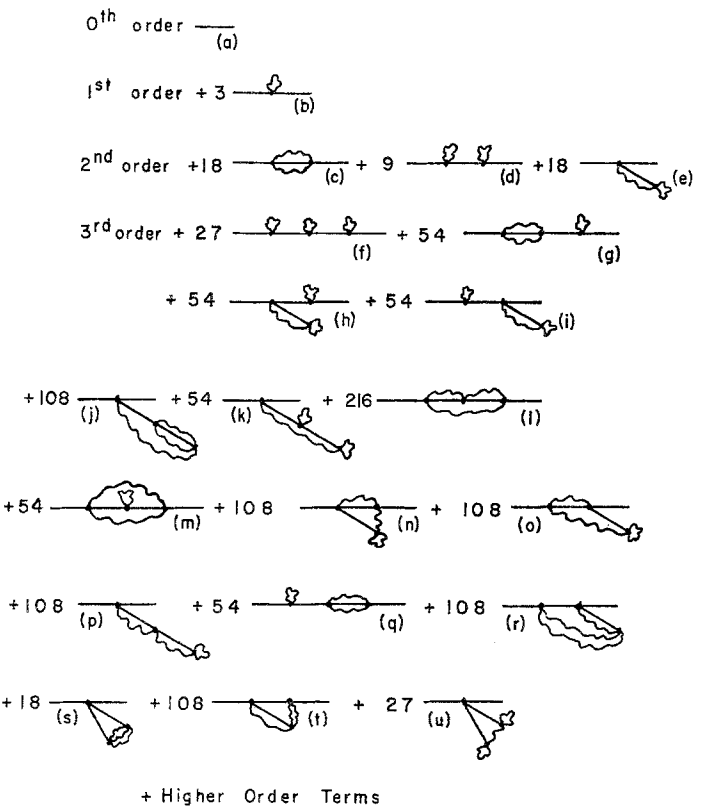
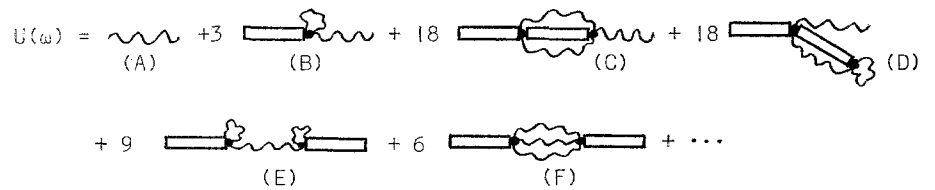


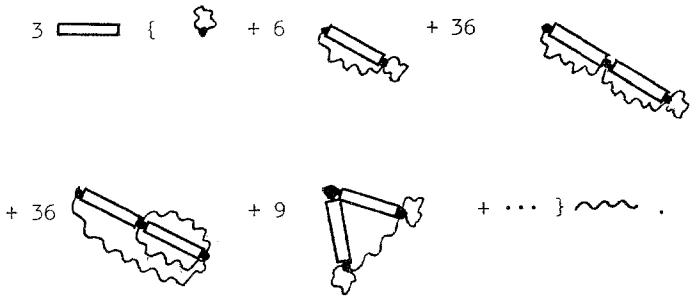
Fig. 5. Diagram for the generalized Green's function to third order in β.

We collect all diagrams which can be formed by replacing one given simple Green's function by all the elements in Fig. 5 and replace this collection by a single diagram with a generalized Green's function as in the example above. We start with the lowest-order diagrams in Fig. 4 and work to higher-order diagrams. Equation (27) becomes, keeping all two-vertex diagrams,



+ complex conjugates of all unsymmetric diagrams. (29)

In order to consolidate further, consider diagrams (*B*), (*D*) of (29) along with appropriate higher-order terms:



(30)

The last three terms in curly brackets in (30) come from three-vertex diagrams.

In (25) and (26) we factored out a set of diagrams and gave the set a new name. Here the factored set of diagrams can be identified with a quantity already defined. Write $U(\omega) \equiv$  . Then the terms in the curly brackets are the same as we would get by putting $U(\omega)$ as expressed by (29) into the summing operation represented by a closed loop, i.e.,

 $\equiv \sum_{\omega'} U(\omega') \equiv \langle x^2 \rangle$

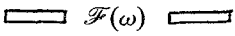
We can consolidate the group of terms in series (30) into one term,

We continue this process by forming groups starting with (*C*) and (*E*) and appropriate higher-order terms, etc.

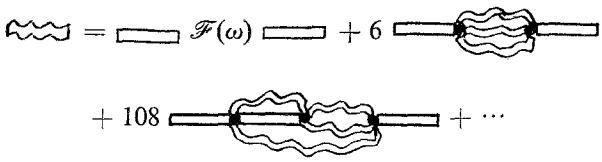
At each order (in β) there remain a small number of terms which cannot be consolidated into groups with lower-order terms. We shall refer to these as "irreducible terms"[22] [e.g., Fig. 4(*g*)].

[22] We shall also refer to consolidated terms which cannot be consolidated further as "irreducible."

It can be shown that the sum of all diagrams in Fig. 4 which either can be cut in two by severing only one wavy line, or begin or end with a wavy line [e.g., (*a*), (*b*), (*c*)] is equal to $S'(\omega)\,\mathscr{F}(\omega)\,S'(-\omega)$, where $\mathscr{F}(\omega) \equiv \langle g(\omega)\,g^*(\omega)\rangle$ is the spectrum of the forcing function.[23] Diagrammatically $S'(\omega)\,\mathscr{F}(\omega)\,S'(-\omega)$ is



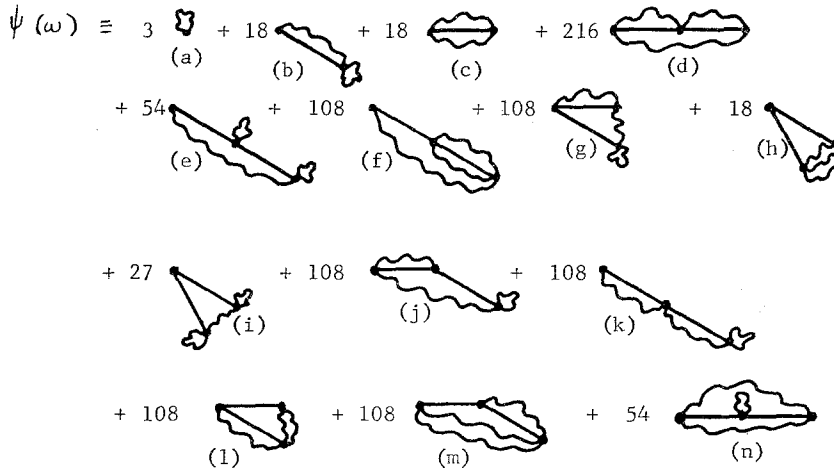
Completing the consolidation of (29), we now have



+ complex conjugates of the unsymmetric terms. (31)

Next we turn to the series for the generalized Green's function $S'(\omega)$, given by Eq. (28) and Fig. 5. In the figure, notice that diagram (*g*) can be constructed by replacing the rightmost simple Green's function in (*c*) by (*b*), and (*d*) can be constructed by replacing the rightmost simple Green's function in (*b*) by (*b*) itself. Any diagram which can be divided into two parts by cutting one internal[23] simple Green's function can be constructed by combining lower-order elements.

This suggests that we define an auxiliary function $\Psi(\omega)$ (say) as what remains if we set aside in Fig. 5 all diagrams which can be cut in two by cutting one internal simple Green's function, and remove each first and last simple Green's function from the remainder:



+ higher-order terms. (32)

[23] By internal simple Green's function we mean any simple Green's function except the first and last on a diagram.

As in series (30), we can group diagrams (*a*), (*b*), (*e*), (*f*), (*h*) and (*i*) of series (32) (plus higher-order terms) and replace the group by

3 

Similarly, diagrams (*c*) and (*g*) of series (32) (and higher-order terms) can be grouped and replaced by

18 

This diagram is the first of a group that can be consolidated into

18 

This process begins a series for Ψ:

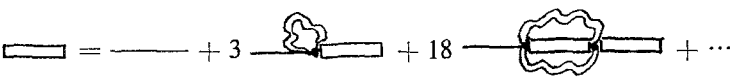
$$\Psi(\omega) = 3 \quad + 18 \quad + \cdots \tag{33}$$


A series for the generalized Green's function, S', can be obtained from the following equation:

$$S'(\omega) = S(\omega) + S(\omega)\,\Psi(\omega)\,S'(\omega). \tag{34}$$

If we solve this iteratively[24] we get all the diagrams in Fig. 5. This reproduces the series for $S'(\omega)$ in Eq. (28).

Use of (33) and (34) gives the diagrammatic consolidated equation for S':



(35)

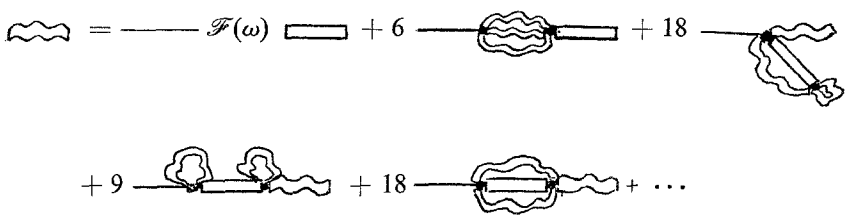
With Eq. (31), this gives two simultaneous equations for the two unknown functions $U(\omega)$ and $S'(\omega)$. It is clear from the way these equations were formed that no other unknown functions appear in the higher-order terms, so Eqs. (31) and (35) are determinate for U and S'. Determinate truncations are considered in Section 4.

Kraichnan's approach to this problem is somewhat different although the final results are similar. He would start with Eq. (7) and multiply by $u(-\omega)$ and average to get [using Eqs. (8) and (9)]

$$\langle u(\omega)\,u(-\omega)\rangle = -\beta S(\omega) \sum_{(1,2,3|0)} \langle u(\omega_1)\,u(\omega_2)\,u(\omega_3)\,u(-\omega)\rangle + S(\omega)\langle u_0(\omega)\,u(-\omega)\rangle.$$

[24] As a first approximation, let $S'(\omega) = S(\omega)$. Put this into the righthand side of Eq. (34) to get a second approximation, $S'(\omega) = S(\omega) + S(\omega)\Psi(\omega)S(\omega)$. Put this second approximation into the righthand side to get a third approximation. Continue this process *ad infinitum*.

He would then expand $\sum_{(1,2,3\mid 0)} \langle u(\omega_1)\, u(\omega_2)\, u(\omega_3)\, u(-\omega)\rangle$ and $\langle u_0(\omega)\, u(-\omega)\rangle$ in series in β using series (10). He would then consolidate in the above manner and finally get Eq. (35) for $S'(\omega)$ and

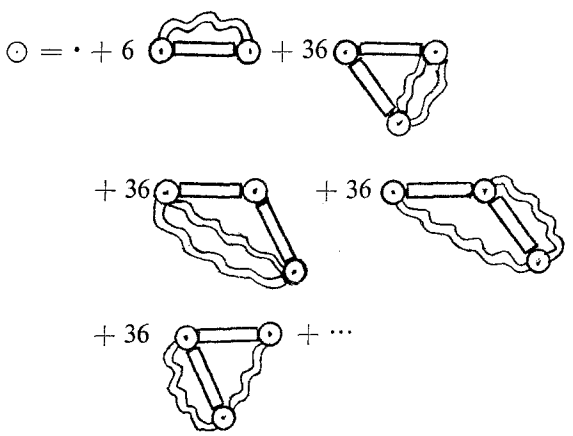


(36)

for $U(\omega)$. Equations (35) and (36), each truncated after two terms, correspond to the "direct-interaction approximation" to Eq. (1).

Next we consider an additional consolidation which adds another unknown, the generalized vertex operator, and requires an additional equation. The generalized vertex operator is defined as the sum of the elements of diagrams in Fig. 4 which have the same properties as a simple vertex, i.e., an element must have (1) four elements entering, (2) all vertices connected by a continuous solid line, and (3) rule 2 (Section 3.3) obeyed for the entire operator. These diagrams are shown to third order in Fig. 6.

Following a consolidation process much like that which led to Eq. (31) and Eq. (35), we get a consolidated equation for this operator:



(37)

We use the letter $\Gamma(\omega, \omega', \omega'')$ for ⊙ where appropriate. In general, this is a function of three variables.

Having identified a new consolidated operator, we look back at our two equations for U and S' to see how they can be consolidated further.

We would like to group the second and third terms on the right-hand side of (31) plus higher-order terms so that we can factor out the terms in Fig. 6 and

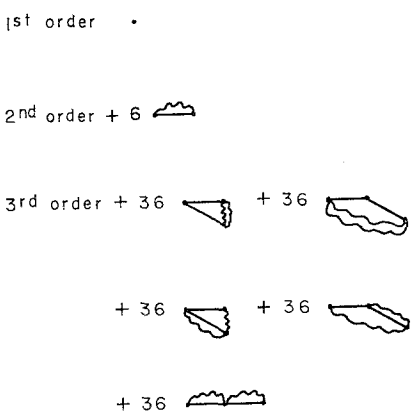
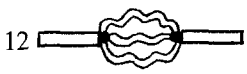
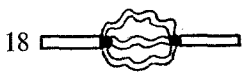


Fig. 6. Diagram for the generalized vertex operator to third order in β.

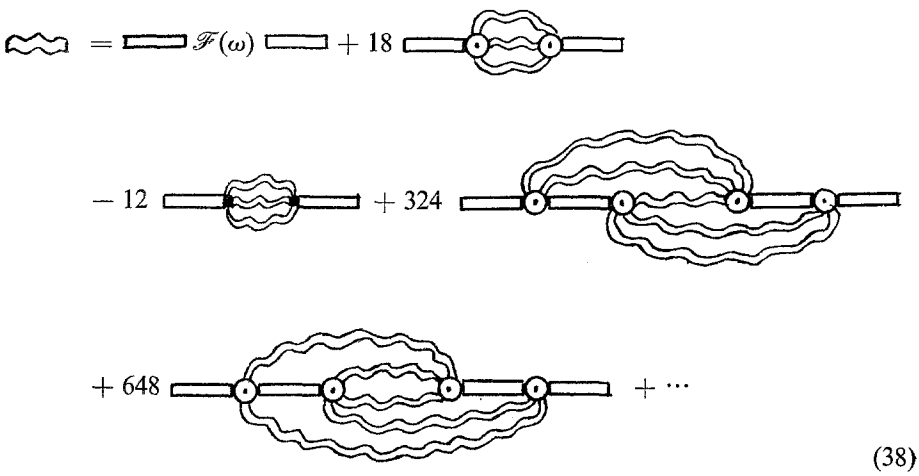
replace them by the generalized vertex operator. Because of the symmetry in the second term on the right of (31), we must add and subtract



Then the third term (plus higher-order terms) can be grouped with

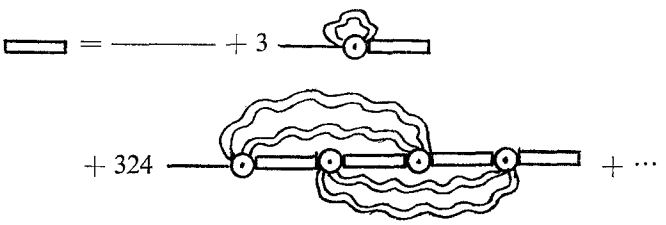


and consolidated. This consolidates (31) to



(38)

All terms in this equation are real (symmetric). Similarly, (35) can be consolidated to



(39)

Together with (37) these give three equations for the three unknown functions U, S' and Γ.

There are no three-vertex diagrams in (38) and no diagrams with two or three vertices in (39).

The next section covers some truncations of the two-equation set, (31) and (35), the "Kraichnan equations," and of the three-equation set (37), (38), (39), the "Kraichnan–Wyld equations" (K–W equations). These will provide integral equations whose solutions may approximate the statistical properties of $x(t)$.

4. TRUNCATION OF THE CONSOLIDATED EQUATIONS

4.1. Introduction

In this section we consider a number of truncations of the K–W equations, (37), (38), and (39), for the Duffing equation. We also discuss truncations of the consolidated set represented by (31) and (35), the "Kraichnan equations."

We shall see that, as far as this calculation was carried for the K–W equations, the higher the "pseudo-order"[25] of the truncation, the better the results. That is, for given α and βD[26] higher truncations give more accurate estimates of the spectrum and the mean-square displacement. Each higher-pseudo-order truncation considered keeps all terms kept by lower-pseudo-order truncations considered.

For the Kraichnan equations (Sections 4.6 and 4.7) one lower-pseudo-order truncation gives better results for the mean-square displacement than a truncation of higher pseudo-order. However, the higher-order truncation gives better estimates of the spectrum, especially at higher frequencies, for some values of $\beta D/2\alpha$.

4.2. The β-Expansion

Before truncating the consolidated equations, we consider truncation of the traditional β-expansion for the spectrum function. We carry the β expansion to second order in β to compare with "nominal second-order" (cf. Section 4.3) truncations of the consolidated equations.

[25] Cf. Section 4.3 for definition of pseudo-order.

[26] D is characteristic of the spectral density function of the random forcing function. For white noise, D is the spectral density.

From Fig. 4 we select all diagrams through the second order. The analytic expression for the spectrum function $U(\omega)$ to second order in β is (in the limit as $T \to \infty$)

$$
\begin{aligned}
U(\omega) = {} & U_0(\omega) - 3\beta M[S(\omega) + S(-\omega)]\, U_0(\omega) \\
& + 9\beta^2 M^2[S(\omega)\, S(\omega) + S(-\omega)\, S(-\omega)]\, U_0(\omega) \\
& + 18\beta^2 \frac{S(\omega)\, U_0(\omega)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_0(\omega - \omega')\, U_0(\omega' - \omega'')\, S(\omega'')\, d\omega'\, d\omega'' \\
& + 18\beta^2 \frac{S(\omega)\, U_0(\omega)\, M}{2\pi} \int_{-\infty}^{\infty} U_0(\omega - \omega')\, S(\omega')\, d\omega' \\
& + 9\beta^2 M^2 S(\omega)\, S(-\omega)\, U_0(\omega) \\
& + 6\beta^2 \frac{S(\omega)\, S(-\omega)}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_0(\omega - \omega')\, U_0(\omega' - \omega'')\, U_0(\omega'')\, d\omega'\, d\omega'' \\
& + \text{complex conjugates of complex terms}^{27} \qquad (40)
\end{aligned}
$$

where $M = 1/2\pi \int_{-\infty}^{\infty} U(\omega)\, d\omega$ and is equal to the mean-square displacement $\langle x^2 \rangle$.

$S(\omega)$ is the simple Green's function [Eq. (9)] and $U_0(\omega)$ is the linear ($\beta = 0$) response spectrum of Eq. (1): $U_0(\omega) = \langle u_0(\omega)\, u_0(-\omega) \rangle$ where $u_0(\omega)$ is given by (8).

These equations were evaluated using a digital computer. The mean-square displacement is the integral of $U(\omega)/2\pi$. $\beta\langle x^2 \rangle/2$ is plotted against $\beta D/2\alpha$ in Fig. 1. The solid line represents the Fokker–Planck (theoretical) values (Section 2.1).

Figure 1 shows that this traditional approximation yields good results for the mean-square displacement for $\beta D/2\alpha \lesssim 0.075$, $\alpha = 2.0$. For $\beta D/2\alpha \approx 0.5$, $\alpha = 0.5$, the spectrum computed by this method develops negative regions, a physical impossibility.

4.3. The Question of Order for the Sets of Consolidated Equations

In Section 3.4 we developed a traditional β expansion for the response spectrum. This expansion is shown, in part, in Fig. 4. There is no trouble defining what we mean by "order"—we define the order of a term as the exponent of β in that term.

In the consolidated equations, the question of "order" is not nearly so clear. Each term except the first in the equations for the generalized Green's function and generalized vertex operator in the consolidated equations represents an infinite subset of terms in the traditional β expansion, hence terms of all orders in β. Let us define the lowest-order (in β) member[28] of this subset as the "fundamental member" of this subset. Then the consolidated term contains terms of all orders in β equal to and higher than the order of its fundamental member. Thus a reasonable definition of the nominal order of a consolidated term would be the order of its fundamental member.

[27] Note that $U(\omega)$ and $U_0(\omega)$ are real while S is a complex function.

[28] By "member" we mean a diagram in the set of diagrams subsumed in a consolidated diagram.

To define an order of truncation of either the K–W or the Kraichnan consolidated equations is a little more difficult. This difficulty arises because we should say something about the distribution of higher-order terms kept. For example, if we truncate both sets of equations, keeping all second-nominal-order terms in both, the K–W equations contain many more of the higher order (in β) terms than do the Kraichnan equations.

Therefore, when referring to truncations of consolidated equations, we need a minimum of two numbers. The first refers to the nominal order of the first irreducible term [cf. equation (31)] in the equation for $U(\omega)$ neglected, minus one. The second number comes from counting the terms in the β expansion with the same order as the fundamental member of the first neglected consolidated term. It is equal to the ratio of the number of terms in the β expansion at this order which are included in the consolidated terms kept, to the total number of terms (in the β expansion).

An equivalent way of defining the first number is to take it equal to the highest order (in β) of the traditional β expansion, all of whose terms are subsumed in the consolidated terms kept after truncation.

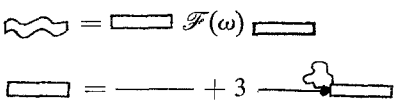
We shall call these two numbers the pseudo-order of the truncation (e.g., the truncation of the K–W consolidated equations, keeping all second-nominal-order terms, has pseudo-order 3-321/323; while the Kraichnan equations, truncated keeping all second-nominal-order terms, have pseudo-order 2-36/44).

It must be emphasized that the nominal order and the pseudo-order are merely operational information. There is no *a priori* reason to expect that a truncation of higher pseudo-order gives a better approximation than one of lower pseudo-order.

4.4. The First Kraichnan–Wyld Approximation

The first truncation we consider is of pseudo-order 1-5/8. One reason for including it is the results have a simple interpretation.

Consider the following truncation of the K–W equations or the Kraichnan equations (to this pseudo-order they give the same results).



(41)

We call this the "first K–W approximation."

We can solve the second equation in (41) for $S'(\omega)$, the generalized Green's function,

$$S'(\omega) = S(\omega) - 3S(\omega)\,\beta M S'(\omega) \tag{42}$$

where

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)\,d\omega$$

is the mean-square displacement $\langle x^2 \rangle$, and

$$S(\omega) = 1/(1 - \omega^2 + i\alpha\omega)$$

From (42),

$$S'(\omega) = \frac{S(\omega)}{1 + 3\beta M S(\omega)}$$

so that

$$S'(\omega) = \frac{1}{(1 + 3\beta M) - \omega^2 + i\alpha\omega} \tag{43}$$

The analytic expression for the first equation in (41) then becomes,

$$U(\omega) = \frac{\mathscr{F}(\omega)}{(1 + 3\beta M - \omega^2)^2 + \alpha^2\omega^2} \tag{44}$$

Notice that $S'(\omega)$ is the Green's function of a linear harmonic oscillator whose damping coefficient is α and whose spring stiffness is $1 + 3\beta M$.

In configuration (i.e., "physical") space, this is an exact solution of

$$\ddot{x} + \alpha\dot{x} + x + 3\beta\langle x^2 \rangle x = f(t) \tag{45}$$

where $\langle x^2 \rangle$ is the mean-square response of (45).

Assume that $\mathscr{F}(\omega) = D$, a constant,[29] in (44), i.e., that $f(t)$ is white noise. Divide by 2π and integrate $U(\omega)$ over all frequencies:

$$M = \frac{D}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^4 + (\alpha^2 - 2 - 6\beta M)\omega^2 + (1 + 3\beta M)^2} \tag{46}$$

Integration yields

$$M = -\frac{1}{6\beta} \pm \frac{1}{\beta}\sqrt{\frac{1}{36} + \frac{\beta D}{6\alpha}}$$

Because $M > 0$, the positive root is the physical one:

$$\frac{\beta M}{2} = -\frac{1}{12} + \frac{1}{12}\sqrt{1 + 12\left(\frac{\beta D}{2\alpha}\right)}. \tag{47}$$

These results could have been gotten directly from (1) by the method of equivalent linearization.[24] In this method the linear equation is assigned a spring stiffness equal to some average of the spring stiffness of the nonlinear system.

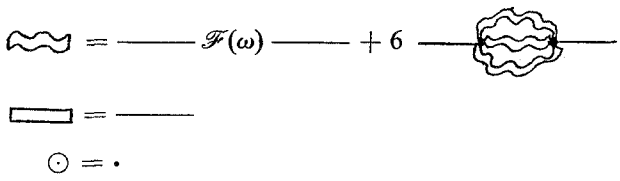
The result (47) is plotted in Fig. 1.

[29] Practically, this means $\mathscr{F}(\omega)$ is constant up to a frequency much higher than any other frequency of interest in the system. [18]

4.5. The Linear Green's-Function Approximation

The next truncation we consider does not keep all second-nominal-order terms in the consolidated equations. This truncation is interesting because it can be shown at least to have positive definite energy, provided that we choose a positive definite initial guess in our iterative solution.

From Eqs. (37), (38) and (39) we keep the following terms



(48)

We shall refer to this truncation as the linear Green's-function approximation because $S'(\omega)$ is replaced by $S(\omega)$. Analytically[30] these equations correspond to

$$U(\omega) = S(+\omega)\left\{\mathscr{F}(\omega) + \frac{6\beta^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[U(\omega-\omega') \times U(\omega'-\omega'')\,U(\omega'')]\,d\omega'\,d\omega''\right\} S(-\omega) \tag{49}$$

$\mathscr{F}(\omega)$ must be positive-definite in order to be realizable. If the initial guess at the solution to (49)[31] is a positive definite even function, it follows that the first approximation, and hence each succeeding one, must be positive definite.

This truncation keeps only a relatively small fraction of even-order terms (but some terms of *all* even orders) in β. Its pseudo-order is 0-0/2. As we can see from Fig. 1, it gives fairly good results for the mean square in the range $\beta D/2\alpha \lesssim 0.1$. For larger values of $\beta D/2\alpha$, it deviates more and more. For $\beta D/2\alpha \approx 0.26$, the iteration scheme did not converge even when the spectrum for the same α and $\beta D/2\alpha = 0.25$ was used as the initial guess.

All the equations we get for each truncation require care in solving. The iteration scheme used to solve these equations did not, in general converge for arbitrary initial guess. This is discussed elsewhere.[23]

The spectrum function was calculated using Eq. (49) for $\alpha = 0.5$ and $\alpha = 2.0$ and for various values of βD.

4.6. The Quasilinear Green's-Function Approximation

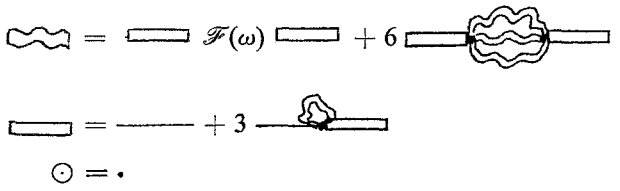
The next truncation we consider resembles the "cumulant discard approximation." This approximation is not a natural truncation of the equations (37), (38) and (39), since it drops second-nominal-order terms in the generalized Green's

[30] Notice that going from the diagrams to the analytic expressions is exactly the same as in Section 3.4 except that we deal with U and S' instead of U_0 and S where applicable.

[31] To be substituted into the integral as the first step in the iteration (or "successive-approximation") method of solution.

function and the generalized vertex operator, while keeping one in the spectrum equation.

The approximation considered is



(50)

We refer to this truncation as the quasilinear Green's-function approximation because, if we are given the mean-square displacement, the equation for the Green's function is linear. Its pseudo-order is 1-6/8. Analytically this is

$$U(\omega) = S'(\omega)\left\{\mathscr{F}(\omega) + \frac{6\beta^2}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega - \omega') \times U(\omega' - \omega'')\, U(\omega'')\right\} S'(-\omega)$$

$$S'(\omega) = S(\omega) - 3\beta M S(\omega)\, S'(\omega) \tag{51}$$

where $M = 1/2\pi \int_{-\infty}^{\infty} U(\omega)\, d\omega = \langle x^2 \rangle$.

It turns out that the cumulant-discard-approximation equations are the following:

$$U(\omega) = S'(\omega)\,\mathscr{F}(\omega)\,S'(-\omega) - \frac{6\beta^2}{(2\pi)^2} S'(\omega)\, S'(-\omega) \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega - \omega')\, U(\omega' - \omega'')\, U(\omega'') \tag{52}$$

where $S'(\omega) = S(\omega) - 3\beta M S(\omega)\, S'(\omega)$.

The second (S') equations in (51) and (52) are identical. The difference lies in the sign of the second term in the equation for $U(\omega)$.

Numerical results based on both of these equations indicate that the quasilinear Green's-function truncation of the K–W equations gives much better results for larger values of $\beta D/2\alpha$. The mean-square values predicted are very good over the whole range of $\beta D/2\alpha$ considered here. Larger values of $\beta D/2\alpha$ would have required the development of noniterative techniques for solving the integral equations.

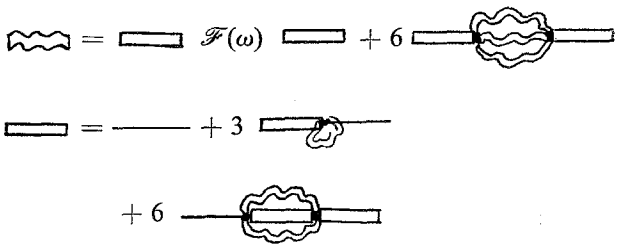
It is important to note that in constructing the consolidated equations, we made no assumptions concerning the cumulants. In fact, Kraichnan[7] points out that for his direct-interaction approximation, some cumulants of all orders are kept. This is true here also. Thus, despite the striking resemblance in the two sets of equations, the assumptions are quite different, so that differing results are to be expected. What is strange is how little difference there is.

4.7. The Direct-Interaction Approximation

It is at this point that the truncations of the equations (37), (38) and (39) and equations (35) and (31) begin to differ. In this section we consider the truncation

of (35) and (31) which keeps all second-nominal-order terms. This corresponds closely to Kraichnan's direct-interaction approximation.[32]

The diagrammatic representation is



(53)

This truncation includes all diagrams to the second order in the β expansion. It also includes terms of all order in β. Its pseudo-order is 2-36/44.

Analytically, equation (53) represents

$$U(\omega) = S'(\omega)\,\mathscr{F}(\omega)\,S'(-\omega) + \frac{6\beta^2 S'(\omega)\,S'(-\omega)}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega-\omega')\,U(\omega'-\omega'')\,U(\omega'')$$

$$S'(\omega) = S(\omega) - 3\beta M S(\omega)\,S'(\omega) + \frac{6\beta^2 S(\omega)\,S'(\omega)}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega-\omega')\,U(\omega'-\omega'')\,S'(\omega'') \tag{54}$$

It follows as in Section 4.5 that the power spectrum is always positive or zero.

Equation (54) can be converted to the D.I.A. [cf. Equation (36)] by replacement of $S'(\omega)$ by $S(\omega)$ while not changing $S'(-\omega)$, and addition of the appropriate second-order terms. With this replacement it is no longer obvious that the spectrum is nonnegative-definite. In the case of the Navier–Stokes equations, it has been shown that the spectrum is nonnegative.[13]

Equation (54) gives good values for the mean square for $\beta D/2\alpha \lesssim 0.25$. For larger values of $\beta D/2\alpha$, $\beta\langle x^2\rangle/2$ grows faster than the exact values. By $\beta D/2\alpha \approx 0.75$,

$$\frac{\langle x^2\rangle_{\text{approx}} - \langle x^2\rangle_{\text{exact}}}{\langle x^2\rangle_{\text{exact}}} \approx 0.30.$$

These results are not as good as those given by the truncation described in the previous section. That success may have been just accidental.

The spectrum agrees well with experiment in regions where the mean square is within a few percent of the exact value (cf. Figs. 7–9).

[32] We call this the "direct interaction approximation" even though it differs slightly from Kraichnan's D.I.A. in the diagrams kept from the β series at fourth and higher orders. Since both (53) and Kraichnan's D.I.A. omit the same third-order (in β) terms, one would not expect the results to differ greatly.

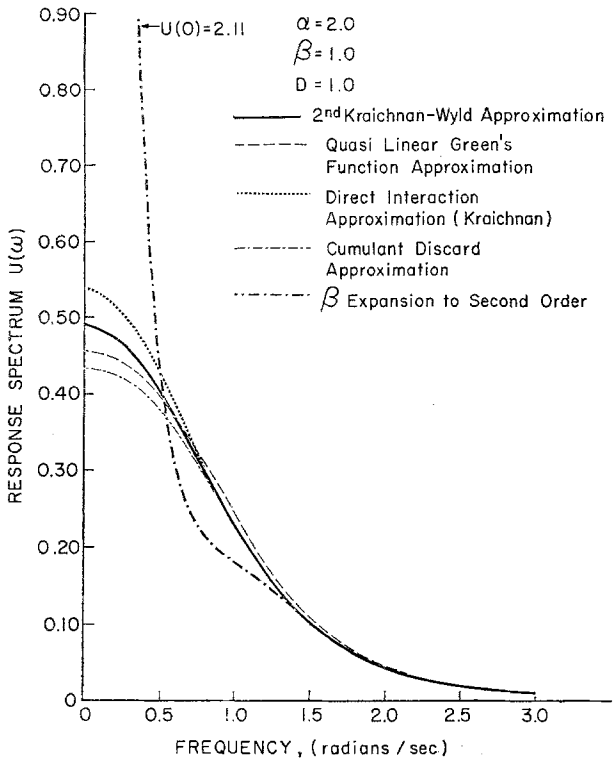
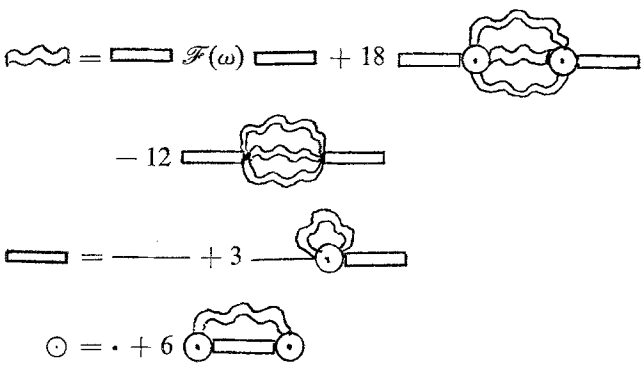


Fig. 7. Response spectrum for relatively high damping.

4.8. The Second Kraichnan–Wyld Approximation

The truncation of (37), (38), and (39), in which we keep all second-nominal-order terms, is quite different from the truncation in the last section. This truncation is



(55)

We call this the second K–W approximation. These equations keep all third-order terms of the β expansion. The pseudo-order is 3-321/323.

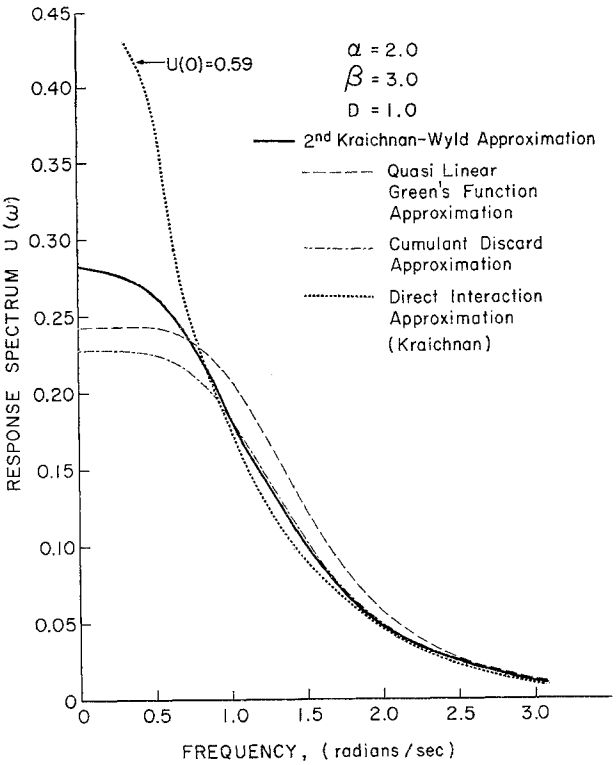


Fig. 8. Response spectrum for relatively high damping.

Analytically,

$$U(\omega) = S'(\omega)\,\mathscr{F}(\omega)\,S'(-\omega)$$
$$+ \frac{18S'(\omega)\,S'(-\omega)}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' \Gamma(\omega')\,U(\omega-\omega')\,U(\omega'-\omega'')\,U(\omega'')\,\Gamma(\omega'')$$
$$- \frac{12\beta^2 S'(\omega)\,S'(-\omega)}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega-\omega')\,U(\omega'-\omega'')\,U(\omega'')$$
$$S'(\omega) = S(\omega) + \frac{3S(\omega)\,S'(\omega)}{2\pi}\int_{-\infty}^{\infty} \Gamma(\omega-\omega')\,U(\omega')\,d\omega'$$
$$\Gamma(\omega) = -\beta + \frac{6\Gamma(\omega)\,\Gamma(-\omega)}{2\pi}\int_{-\infty}^{\infty} U(\omega-\omega')\,S'(\omega')\,d\omega' \qquad (56)$$

The form of these equations again assures us of nonnegative definite energy.

The mean-square displacement predicted by Eq. (56) is very good up to $\beta D/2\alpha \approx 0.8$. For $\beta D/2\alpha \gtrsim 0.8$, growing computational-stability problems made calculations difficult.

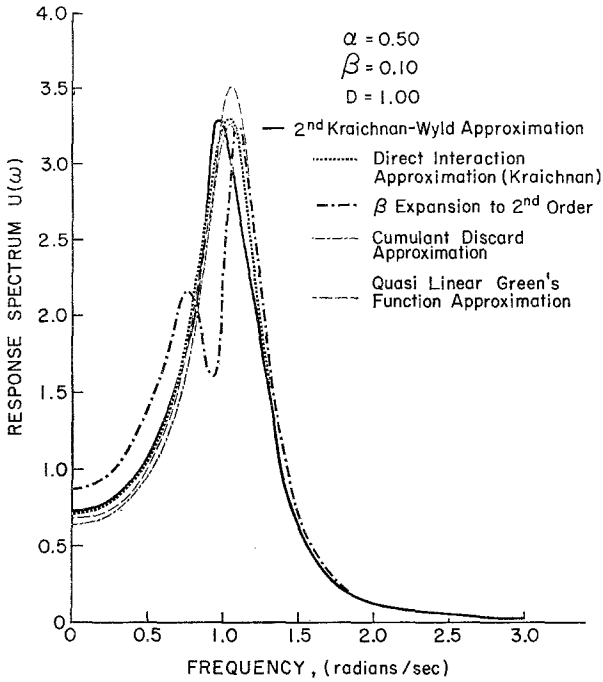


Fig. 9. Response spectrum for relatively low damping.

4.9. The Numerical Results

The response spectra predicted by the truncations discussed in this chapter are shown in Figs. 7–9. As $\beta D/2\alpha \to 0$ ($\alpha = 2.0$ and $\alpha = 0.5$) all of the truncations approach the same (linear) results. The amplitude and phase of a typical modified Green's function are shown in Figs. 13–14.

Figures 10–12 give the second K–W approximation, numerically filtered, with data measured from the analog-computer experiment (Section 2.2).

5. HIGHER MOMENTS

5.1. Preliminary Remarks

In this section it is shown that, with restriction to a Gaussian forcing function, all higher moments for the response of (1) can be expressed in terms of the spectrum function, generalized Green's function, and generalized vertex operator. As an example, we consider a two-time fourth moment (including the "flatness factor") for the solution of equation (1).

5.2. The Fourth Moment of the Response of Eq. (1)

In Section 3, we used the β series for the instantaneous Fourier amplitude field to develop equations for the spectrum function of (1). We can also use this method to develop consolidated expansions for other statistical functions.

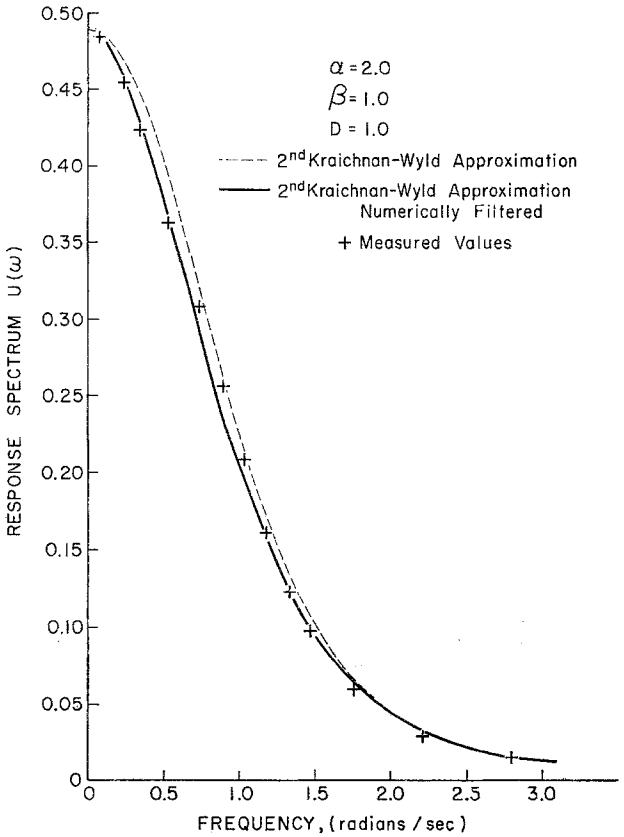


Fig. 10. Measured response spectrum compared with the second K–W approximation for relatively high damping.

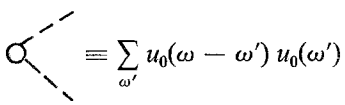
The Fourier transform of

$$F(\tau) = \langle x^2(t)\, x^2(t+\tau)\rangle \tag{57}$$

in the stationary case is

$$\sum_{\omega',\omega''} \langle u(\omega-\omega')\, u(\omega')\, u^*(\omega-\omega'')\, u^*(\omega'')\rangle \tag{58}$$

Define a symbol ○ such that



Assume that $u = u_0 + u_1 + \cdots$ where u_0, u_1, etc., are expressed in terms of the diagrams in Fig. 4.

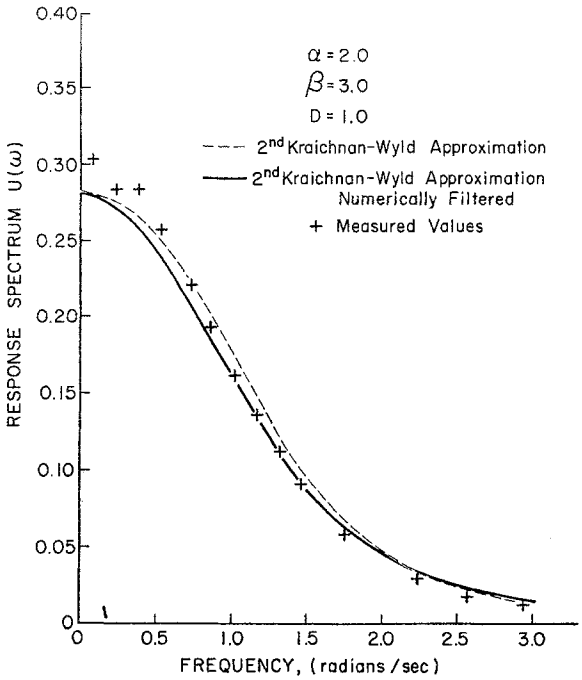


Fig. 11. Measured response spectrum compared with the second K–W approximation for relatively high damping.

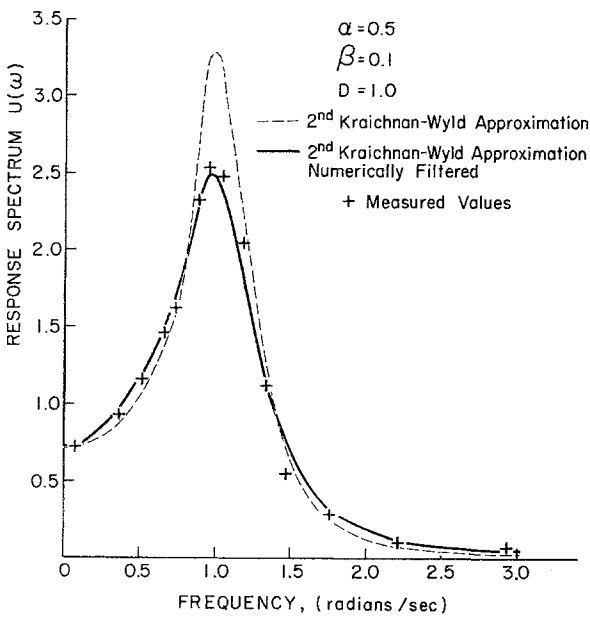


Fig. 12. Measured response spectrum compared with the second K–W approximation for relatively low damping.

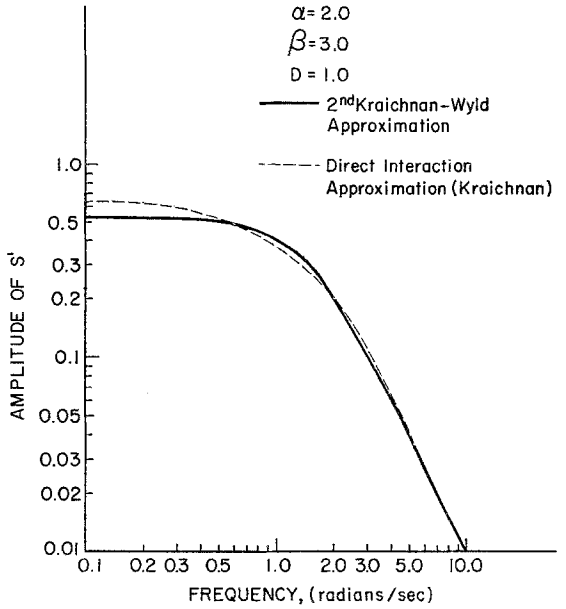


Fig. 13. Amplitude of the generalized Green's function for relatively high damping.

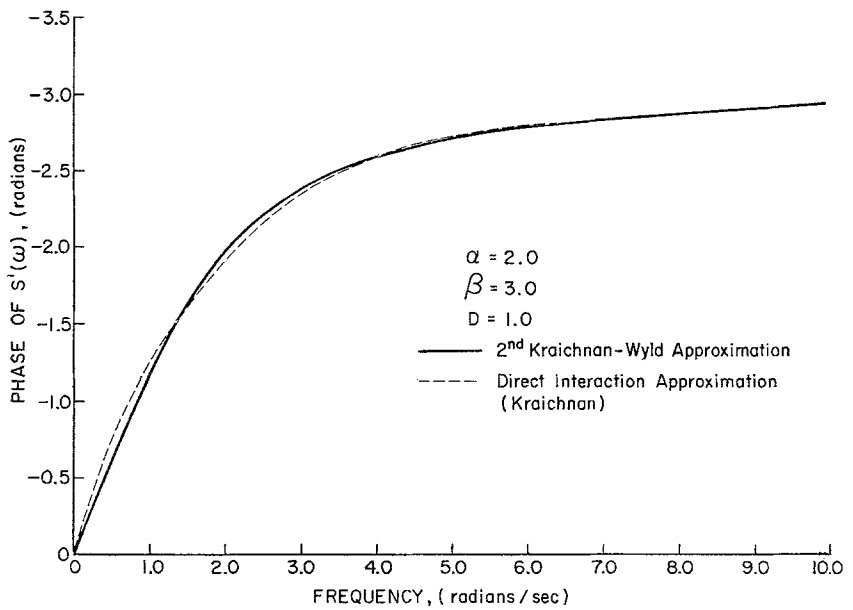
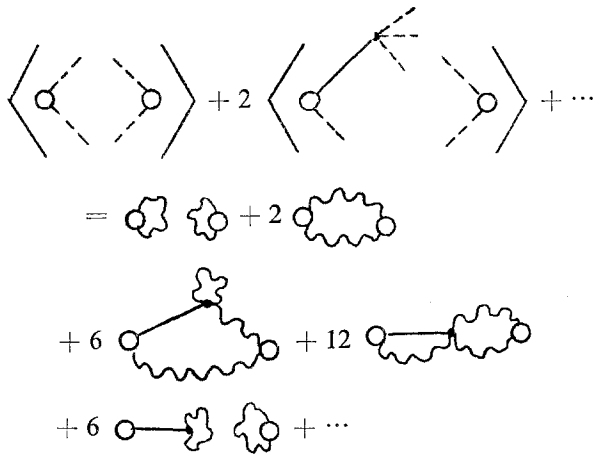


Fig. 14. Phase of the generalized Green's function for relatively high damping.

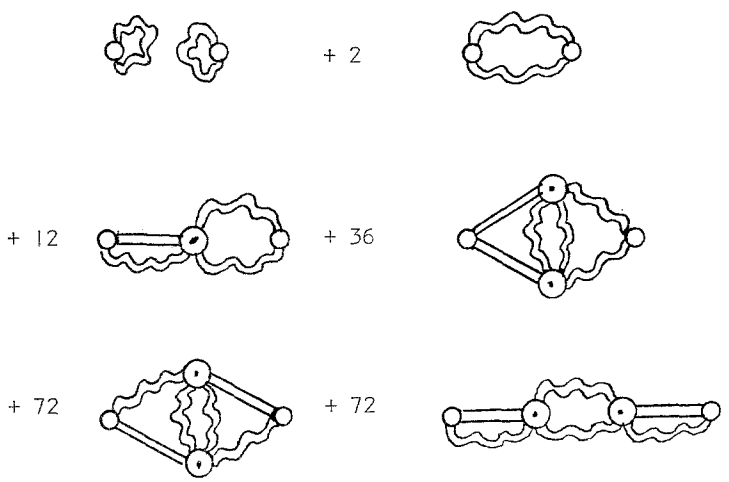
In terms of diagrams, (58) is



+ complex conjugates of unsymmetric diagrams (59)

These diagrams, shown to second order in Fig. 15, contain elements that are identifiable with diagrams from Figs. 4, 5, and 6. This allows us to group and consolidate terms in (59) in the same way as done previously.

To second nominal order we get the fourth moment equal to



+ ⋯ + complex conjugates of unsymmetric terms (60)

The flatness factor, $\langle x^4\rangle/\langle x^2\rangle^2$, is the sum of (58) with respect to ω [which is the $\tau \to 0$ limit of (57)], divided by square of the mean-square displacement.

The first term in (60) is just

$$\left\{\sum_{\omega} U(\omega)\right\}\left\{\sum_{\omega'} U(\omega')\right\} = M^2,$$

where M is the mean-square displacement.

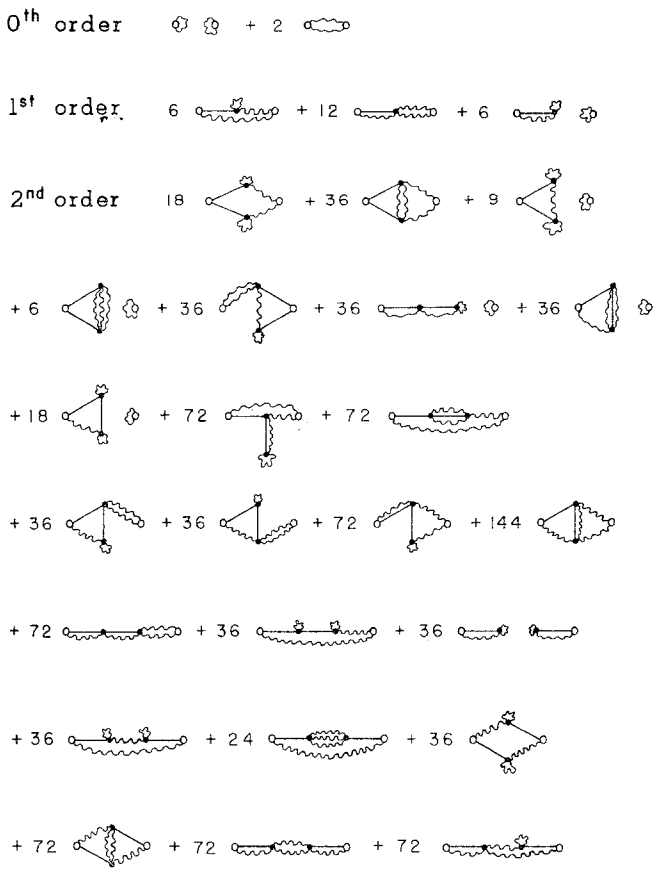


Fig. 15. Diagrams for the fourth moment considered in Section 5.2.

The second term is

$$2\sum_{\omega'} U(\omega')\, U(\omega - \omega')$$

Summing over ω gives

$$2\left\{\sum_{\omega'} U(\omega')\right\}\left\{\sum_{\omega} U(\omega)\right\} = 2M^2$$

Thus the first two terms contribute the (Gaussian) value 3.0 to the flatness factor.

One of the difficulties in going from the diagrams to the analytic expressions is the interpretation of the generalized vertex operator. In the last four terms explicitly shown in (60) the generalized vertex operator can be interpreted in two different ways. For example, consider the second-order contribution to the generalized vertex operator and the third term in (60). This term has consolidated in it

Thus when going to the analytic forms we must interpret the generalized vertex operator both ways, taking care to subtract out excess contributions made by the simple vertex, as we did in (38).

The second of these interpretations yields

$$12 \sum_{\omega',\omega''} U(\omega - \omega')\, S'(\omega')\, \Gamma(\omega)\, U(\omega - \omega'')\, U\omega'').$$

In the limit as the time interval (period) goes to infinity, the contribution to the flatness factor is

$$\frac{24}{2\pi M^2} \int_{-\infty}^{\infty} d\omega (U^*S)(\omega)(U^*U)(\omega)\, \Gamma(\omega) \tag{61}$$

where the complex conjugate has been included, and the asterisk (*) indicates a convolution.

The rest of the terms in (60) can similarly be interpreted. This example illustrates that once we have consolidated equations for the spectrum, generalized Green's function and generalized vertex operator, we can find all higher moments in terms of these functions.

Tentative numerical results (cf. Fig. 2) indicate that truncating (59) after the third term (referred to as the "first truncation") gives better results than keeping the six terms explicitly shown (the "second truncation"). This is not surprising since the consolidation considered here leads directly to an equation for the second moment. The three functions calculated in Sections 3 and 4 would not be expected to contain much information about higher moments.

These results are tentative because there is no way to check them directly. As a partial check we used the fourth moment $\langle x^3(t)\, x(t + \tau)\rangle$ to get the flatness factor, and the results were the same. More work on this will be necessary before a firm conclusion can be drawn.

6. CONCLUDING REMARKS

Kraichnan[12] refers to the consolidated equations as consolidated "expansions." This terminology could be misleading because the word "expansion" usually refers to situations in which the "unknown" is presented in isolation, equal to a series of "known" quantities. In contrast, the consolidated equations are integral equations with the unknown function appearing in all the terms except the first on the "right-hand side." The expression "consolidated expansion" applies very well, on the other hand, to the expressions for the higher moments discussed in Section 5.

The numerical results reported here suggest that this distinction between consolidated equations and consolidated expansions may be relevant. The keeping of higher-nominal-order terms (in the sense of Section 4.3) in the consolidated equations improved the results, whereas the keeping of higher-nominal-order terms in the consolidated expansion did not.

Another question is how the solutions of these consolidated equations relate to the solution of the original problem. If the consolidated equations are solved

iteratively [in the same manner as (34)], the terms in the iterative solution are identical to the terms in the simple parametric expansion. Thus, if the response spectrum is analytic so that the parametric expansion represents the function, and if the solution of the consolidated equations is unique, then the solution of the consolidated equations is indeed the response spectrum of the original equation. The question of the uniqueness of the solutions of the consolidated equations (or their existence) and the question of the analyticity of the response spectrum for the consolidated equations are still unanswered.

We should also ask how rapidly the solution of the truncations of the consolidated equation approach the solutions of the untruncated consolidated equations as the pseudo-order of truncation is increased. Or do they approach it at all, except in the limit? A partial answer to this question was suggested by the numerical results in Section 4. There the higher pseudo-order truncation (in the sense of Section 4.3), gave a better spectrum. The second K–W approximation gave very good results for the response spectrum in all cases tested.

It was seen in Section 3 that there is no unique way to form consolidated equations. The Kraichnan hierarchy and the K–W hierarchy are both plausible consolidations. One difference was the introduction of an additional generalized operator (hence an additional step of consolidation), the generalized vertex. Are there additional operators which could be introduced to provide further consolidation? The answer to this is apparently negative. Any new generalized operator would have to be devised from elements of consolidated diagrams. There do not seem to be any groups of elements of consolidated diagrams with the same properties, which could be factored out to define a new generalized operator.

In other problems it may be helpful to introduce more than one of each kind of generalized operator,[15] although this did not seem helpful for the Duffing equation.

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